Schur's Theorem in Integer Lattices

Andrew Lott and Vishal Balaji (with faculty advisor Alex Rice)

Millsaps College

March 2023

Definitions

- For $N \in \mathbb{N}$, $[N] = \{1, 2, \ldots, N\}$
- For $r \in \mathbb{N}$, an **r-coloring** is a partition of a set into r pairwise disjoint subsets (color classes).
- A set is **monochromatic** if it is contained in a single color class.
- A **Schur triple** is a solution to $a + b = c$.

Schur's theorem

Here is a 2-coloring of [4]: $\{1, 2, 3, 4\}$.

Notice there are no monochromatic Schur triples. Indeed,

$$1 + 1 = 2, 1 + 2 = 3, 1 + 3 = 4, 2 + 2 = 4.$$

Schur's theorem

But what about 2-colorings of $\{1, 2, 3, 4, 5\}$? It turns that any such coloring yields a monochromatic Schur triple!

Let's try to construct a coloring of $\{1, 2, 3, 4, 5\}$ in red and blue with no monochromatic Schur triple and come to a contradiction.

Schur's theorem

$$\{1,2,3,4,5\}$$

Schur's theorem

$$\{\textcolor{blue}{1},\textcolor{red}{2},3,4,5\}$$

Schur's theorem

$$\{1,2,3,4,5\}$$

Schur's theorem

$$\{1,2,3,4,5\}$$

Schur's theorem

$\{1,2,3,4,5\}$

But what color is left for 3?

$1+3=4$, so 3 is not blue.

$2+3=5$, so 3 is not red.

This is our contradiction!

Schur's theorem

But what about **Schur 4-tuples** ($a + b + c = d$) or **Schur 5-tuples** ($a + b + c + d = e$)?

What if we used three colors (*red*, *green*, *blue*) or four colors (*red*, *green*, *blue*, *orange*)?

Schur's theorem handles Schur tuples of any length and any number of colors.

Schur's theorem

Theorem

Let $r, k \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that every r-coloring of $\{1, 2, \ldots, N\}$ yields a monochromatic Schur k-tuple. In particular, there is a smallest such N, denoted $S(r, k)$.

Our example was $r = 2$ (red and blue) and $k = 3$ (Schur triples, solutions to $a + b = c$). We determined $S(2, 3) = 5$.

The Schur numbers

$$S(3,3) = 14, S(4,3) = 45, S(5,3) = 161$$

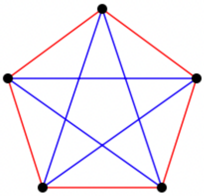
Marjin Heule confirmed the value of $S(5,3) = 161$ with a **SAT solver** in 2018.

Notice the number of 5-colorings of [161] is 5^{161}. This is far more than the number of atoms in the observable universe!

Ramsey's Theorem (1928)

Theorem

For $r, k \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that every r-coloring of the edges of a complete graph on N vertices yields a monochromatic complete subgraph on k vertices. In particular, there is a smallest such N denoted $R_r(k)$.



Schur's Theorem in higher dimensions

Can Schur's theorem be generalized to integer lattices?

We call $x_1, ..., x_k \in \mathbb{N}^d$ a **Schur k-tuple** if $x_1 + \cdots + x_{k-1} = x_k$.

With this definition, the problem boils down to the 1-dimensional case: the smallest N such that $[N]^d$ is guaranteed to contain a monochromatic Schur k-tuple is $S(r, k)$.

1 2 3 4 5 6 7 8 9 10 11 12 13

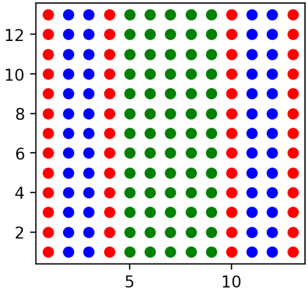


Figure: A 3-coloring of $[13]^2$ with no monochromatic Schur triple:

For $d, r, k \in \mathbb{N}$, any r-coloring of $[S(r,k)]^d$ admits a monochromatic Schur k-tuple on the diagonal $\{(x, \ldots, x) : x \in [S(r,k)]\}$.

Main Theorem

Luckily, a linear independence condition makes the problem more interesting!

Let $k, d \in \mathbb{N}$ with $k \leq d+1$. We call $x_1, ..., x_k \in \mathbb{N}^d$ a **nondegenerate Schur k-tuple** if $x_1, \ldots, x_{k-1}$ are linearly independent and $x_1 + \cdots + x_{k-1} = x_k$.

Theorem

Let $r, d \in \mathbb{N}$. There exists $N \in \mathbb{N}$ such that every r-coloring of $[N]^d$ yields a monochromatic nondegenerate Schur $(d+1)$-tuple.

Proof of main theorem

Fix an r-coloring of the lattice $[R_r(d+1)^d - 1]^d$

For $1 \leq i \leq R_r(d+1)$, let $y_i = (i, i^2, \ldots, i^d)$.

We view the y_i's as the vertices of a complete graph whose edges are given by the differences $y_j - y_i$ for $1 \leq i < j \leq R_r(d+1)$.

We give each each edge $y_i - y_j$ the color of $y_i - y_j$ in the r-coloring of $[R_r(d+1)^d - 1]^d$.

Proof of main theorem (continued)

This is an r-coloring of a complete graph on $R_r(d+1)$ vertices, so we apply Ramsey's theorem!

By definition of $R_r(d+1)$, there exist $y_{i_1}, \ldots, y_{i_{d+1}}$ with $i_1 < \ldots < i_{d+1}$ which induce a monochromatic complete subgraph.

Translating this back to the lattice coloring, we observe $y_{i_{j+1}} - y_{i_j}$ for $1 \le j \le d$ and $y_{i_{d+1}} - y_{i_1}$ are all the same color. Further, they satisfy

$$(y_{i_2} - y_{i_1}) + (y_{i_3} - y_{i_2}) + \cdots + (y_{i_{d+1}} - y_{i_d}) = y_{i_{d+1}} - y_{i_1}.$$

Proof of main theorem (continued)

In addition, $\{y_{i_2} - y_{i_1}, \ldots, y_{i_{d+1}} - y_{i_d}\}$ are linearly independent.

Indeed, the following matrix has non-zero determinant:

$$A = \begin{bmatrix} i_2 - i_1 & i_2^2 - i_1^2 & \cdots & i_2^d - i_1^d \\ i_3 - i_2 & i_3^2 - i_2^2 & \cdots & i_3^d - i_2^d \\ \vdots & \vdots & \ddots & \vdots \\ i_{d+1} - i_d & i_{d+1}^2 - i_d^2 & \cdots & i_{d+1}^d - i_d^d \end{bmatrix}.$$

The determinant can be easily obtained by applying determinant preserving row operations to the $(d+1) \times (d+1)$ Vandermonde matrix

$$V = \begin{bmatrix} 1 & i_1 & i_1^2 & \dots & i_1^d \\ 1 & i_2 & i_2^2 & \dots & i_2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & i_{d+1} & i_{d+1}^2 & \dots & i_{d+1}^d \end{bmatrix},$$

resulting in

$$\begin{bmatrix} 1 & i_1 & i_1^2 & \dots & i_1^d \\ 0 & i_2 - i_1 & i_2^2 - i_1^2 & \dots & i_2^d - i_1^d \\ 0 & i_3 - i_2 & i_3^2 - i_2^2 & \dots & i_3^d - i_2^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & i_{d+1} - i_d & i_{d+1}^2 - i_d^2 & \dots & i_{d+1}^d - i_d^d \end{bmatrix}.$$

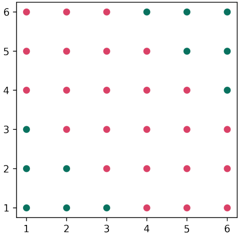


Figure: A 2-coloring of $[6]^2$ with no monochromatic nondegenerate Schur triples.

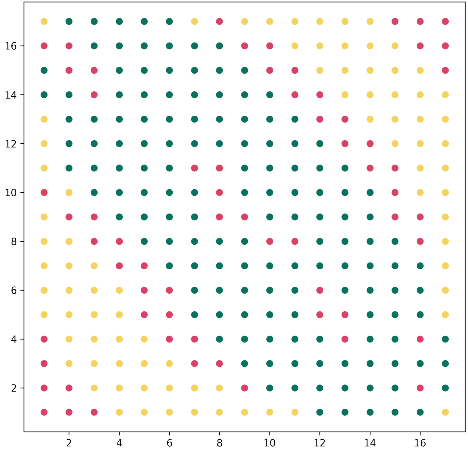


Figure: A 3-coloring of $[17]^2$ with no monochromatic nondegenerate Schur triples.

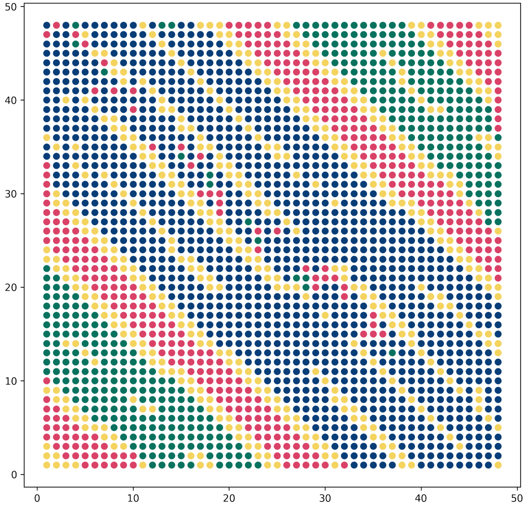


Figure: A 4-coloring of $[48]^2$ with no monochromatic nondegenerate Schur triples.