Polynomial configurations in the primes

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INTEGERS 25

Notation

- $[N] = \{1, 2, \ldots, N\}$
- $\mathbb{P}_N$ is the set of primes $p \leq N$.
- A k-term arithmetic progression (kAP) is a set of integers of the form

$$x,\ x+y,\ x+2y,\ \ldots,\ x+(k-1)y.$$

Theorem (Szemerédi)

Let $k \in \mathbb{N}$ and $\delta > 0$. If N is large enough and $A \subset [N]$ with $|A|/N > \delta$, then A contains a kAP.

- The case $k = 3$ is known as Roth's theorem.
- Szemerédi proved the theorem for $k \geq 4$ in the 70s using very difficult but elementary methods.
- Furstenberg developed an elegant proof via ergodic theory.
- Gowers developed higher order Fourier analysis in order to prove a much better quantitative version of this theorem.

Theorem (Bergelson–Leibman)

Let $p_1, \ldots, p_k \in \mathbb{Z}[y]$ where $p_1(0) = \cdots = p_k(0) = 0$. Let $\delta > 0$. If N is large enough and $A \subset [N]$ with $|A|/N > \delta$, then A contains a polynomial progression of the form

$$x + p_1(y), \ \ldots, \ x + p_k(y).$$

Without the condition that $p_1, \ldots, p_k$ have zero constant terms, we might run into congruence obstructions. For instance, if A is the set of multiples of 3, then $\overline{\delta}(A) = \frac{1}{3}$, but A contains no pattern of the form $x, x + y^2 + 1$, since $y^2 + 1$ is never a multiple of 3.

- As of now, the proofs of the general form of Bergelson-Leibman involve ergodic theory, which is elegant but all quantitative information is lost.
- Sean Prendiville gave a proof yielding quantitative bounds in the case when $p_1, \ldots, p_k$ have the same degree.
- Sarah Peluse developed a technique called degree lowering which allowed her to prove a quantitative version of the theorem in the case when $p_1, \ldots, p_k$ have distinct degrees.

Theorem (Multidimensional Bergelson-Leibman)

Let $p_1, \ldots, p_k \in \mathbb{Z}[y]$ *with* $p_1(0) = \cdots = p_k(0) = 0$ *and* $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{Z}^d$. *Let* $\delta > 0$. *Let* N *be large enough and suppose* $A \subset [N]^d$ *with* $|A|/N^d > \delta$. *Then* A *contains a pattern of the form*

$$\vec{x} + p_1(y)\vec{v}_1, \ \cdots, \ \vec{x} + p_k(y)\vec{v}_k.$$

- As before, the only known proof for the general statement of the theorem relies on ergodic theory and thus gives no quantitative bounds.
- Kuca, Leng, and Kravitz have a result giving quantitative bounds for certain polynomial corners

Theorem (Green-Tao)

Let $\delta > 0$, $k \in \mathbb{N}$. If N is large enough and $A \subset \mathbb{P}_N$ with $|A| \geq \delta|\mathbb{P}_N|$, then A contains a kAP.

- Notice that this is not an immediate consequence of Szemerédi's theorem because the primes have density zero in $\mathbb{N}$.
- The idea of the proof is to use the **transference principle** to model A, which is relatively dense in the primes, by a set B, which is dense in $\mathbb{N}$. Then, by Szemerédi's theorem, B has a kAP, so A also has a kAP.

Theorem (Tao-Ziegler)

Let $\delta > 0$, and $p_1, ..., p_k \in \mathbb{Z}[y]$ with $p_1(0) = \cdots = p_k(0) = 0$. If N is large enough and $A \subset \mathbb{P}_N$ with $|A| \geq \delta|\mathbb{P}_N|$, then A contains polynomial progression of the form $x + p_1(y), ..., x + p_k(y)$, where $y < \log^L(N)$, where L is some larger integer independent of N.

- The condition that $y < \log^L(N)$ says that the progressions are "narrow" in a sense.

Main Result 1

Theorem (L., Magyar, Petridis, Pintz, 2025+)

Let $\delta > 0$, and $p_1, ..., p_k \in \mathbb{Z}[y]$ with $p_1(0) = \cdots = p_k(0) = 0$ and $\vec{v}_1, ..., \vec{v}_k \in \mathbb{Z}^d$. Suppose that $p_1(y)\vec{v}_1, ..., p_k(y)\vec{v}_k$ is "non-degenerate". If N is large enough and $A \subset \mathbb{P}_N^d$ with $|A| \geq \delta |\mathbb{P}_N|^d$, then A contains a pattern of the form

$$\vec{x} + p_1(y)\vec{v}_1, \ \cdots, \ \vec{x} + p_k(y)\vec{v}_k,$$

where $y < \log^L(N)$.

- This is a hybrid of the theorems of Bergelson-Leibman and Tao-Ziegler, with the condition that the polynomial system is "non-degenerate".
- An example of the type of polynomial pattern we cannot handle is the corner. We suspect that it is possible to prove the result for all of the polynomial patterns you get from Bergelson-Leibman.

Theorem (Maynard)

Let $m \in \mathbb{N}$ and suppose k is large enough with respect to m. Let $h_1, ..., h_k$ be an admissible k-tuple. There exist infinitely many $n \in \mathbb{N}$ such that at least $m + 1$ of $n + h_1, ..., n + h_k$ are prime.

- Polymath optimized Maynard's sieve. In the case $m = 1$, they were able to find an admissible 50-tuple which satisfied the conclusion of the theorem. With this, they were able to prove that there is a positive integer $b \leq 246$ such that there are infinitely many prime pairs $p, p + b$.

Main Result 2

Theorem (L., Ponagandla, 2025+)

There exists a positive integer $b \leq 246$ *such that the following holds. Let* $p_1, \ldots, p_k \in \mathbb{Z}[y]$ *with* $p_1(0) = \cdots = p_k(0) = 0$*. There exist infinitely many* $x, y \in \mathbb{Z}$ *such that* $x + p_1(y), \ldots, x + p_k(y)$ *and* $x + p_1(y) + b, \ldots, x + p_k(y) + b$ *are all prime.*

- In other words, there are bounded gaps between the polynomial progressions in the primes.