Notes for UGA NT Seminar Fall 2025

For $N \in \mathbb{N}$, let $D(N)$ denote the size of the largest subset of $\{1, ..., N\}$ containing no two elements differing by $p-1$ with p prime.

Big question: what is the shape of $D(N)$?

Conjecture (Erdos)

$$\lim_{N\to\infty} \frac{D(N)}{N} = 0.$$

Theorem (Sarkozy, 1978) There exists a constant $C > 0$ such that for any integer $N > 2$,

$$D(N) < \frac{CN}{(\log\log N)^2}.$$

Theorem (Ruzsa-Sanders, 2008) There exist constants $C, d > 0$ such that for any integer $N > 2$,

$$D(N) < CNe^{-d(\log N)^{1/4}}.$$

Both Sarkozy and Ruzsa-Sanders relied on "increment methods".

Recently, Ben Green used the van der Corput property to prove the following.

Theorem (Green, 2023) There exist constants $C, d > 0$ such that for any $N \in \mathbb{N}$,

$$D(N) < CN^{1-d}.$$

Theorem (Green, 2023) Assume GRH. Let $\epsilon > 0$. There exists a constant $C_\epsilon > 0$ such that for any $N \in \mathbb{N}$,

$$D(N) < C_\epsilon N^{11/12+\epsilon}.$$

What about a lower bound on $D(N)$?

Theorem (Ruzsa, 1984) There exists a constants $C, d > 0$ such that for any integer $N > 2$,

$$D(N) \geq CN^{d/\log\log N}.$$

Green believes the true shape of $D(N)$ is the same shape as the lower bound from Ruzsa's construction.

Let $\mathbb{F}_q$ be the finite field of order $q = p^k$. Let $\mathbb{F}_q[t]$ be the ring of polynomials over $\mathbb{F}_q$.

For $N \in \mathbb{N}$, let $D_q(N)$ denote the size of the largest subset of $\{f \in \mathbb{F}_q[t] : \deg f \leq N\}$ containing no two elements differing by $P-1$ with P irreducible.

Using Weil's analogue of GRH in $\mathbb{F}_q[t]$, we have the following.

Theorem(Fan, L. 2025) Let $\epsilon > 0$. There exist constants $d_q, D_q, C_{\epsilon,q} > 0$ such that any integer $N > 2$, we have

$$D_q q^{(N+1)\frac{d_q}{\log N}} < D_q(N) < C_{\epsilon,q} q^{(N+1)(11/12+\epsilon)}$$

Function field set-up

Let K be the field of fractions of $\mathbb{F}_q[t]$.

Let $|\cdot|$ be a norm on $\mathbb{F}_q[t]$ defined by $|f| = q^{\deg f}$, where we adopt the convention $\deg 0 = -\infty$, $q^{-\infty} = 0$.

$|\cdot|$ extends naturally to K. Let K_∞ be the completion of K with respect to $|\cdot|$.

Every $x \in K_\infty$ has a Laurent expansion with finitely many positive degree terms:

$$x = \sum_{n<n_0} a_n t^n.$$

with each $a_n \in \mathbb{F}_q$. Let $\mathrm{res}(x)$ denote the coefficient of t^{-1}.

The trace map $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is given by

$$\mathrm{tr}(a) = a + a^p + a^{p^2} + \cdots + a^{p^{k-1}}.$$

We define a complex exponential on K_∞ as follows. For $x \in K_\infty$,

$$e(x) = e^{2\pi i \mathrm{tr}(\mathrm{res}(x))/p}.$$

Lemma(Orthogonality relation) Let $f \in \mathbb{F}_q[t]$. Then

$$\sum_{\deg g<N} e(fg/t^N) = \begin{cases} 0 & \text{if } t^N \nmid f \\ q^N & \text{if } t^N \mid f. \end{cases}$$

Proposition (The van der Corput property) Let $\epsilon > 0$. There exists a constant $C_{\epsilon,q}$ such that the following holds. Let $N \in \mathbb{N}$. There exists a cosine polynomial $T : K_\infty \to \mathbb{R}$ given by

$$T(x) := a_0 + \sum_{\substack{\deg P \leq N \\ P-\text{irreducible}}} a_{P-1} \Re e((P-1)x),$$

where $a_f \in \mathbb{R}$ and

(1) $T(0) = 1$.
(2) $T(x) \geq 0$ for every $x \in K_\infty$.
(3) $0 < a_0 < C_{\epsilon,q} q^{(N+1)(-1/12+\epsilon)}$.

The van der Corput property immediately implies the upper bound in our main theorem. Indeed, let $N \in \mathbb{N}$. Let T and $C_{\epsilon,q}$ be as above.

Let $A \subset \{f \in \mathbb{F}_q[t] : \deg f \leq N\}$ have no two elements differing by $P - 1$ with P irreducible.

$$|A|^2 \leq \sum_{\deg r \leq N} \left| \sum_{\deg f \leq N} 1_A(f) e(rf/t^{N+1}) \right|^2 T(r/t^{N+1}) \leq a_0 q^{N+1} |A|$$

The lower bound comes from properties (1) and (2). The upper bound comes from applying the orthogonality relation twice.

In total, this gives

$$|A| \leq q^{N+1} a_0 < C_{\epsilon,q} q^{(N+1)(11/12+\epsilon)}$$